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BPS branes from baryons

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Abstract: We elucidate the relationship between supersymmetric $D3$-branes and chiral baryonic operators in the AdS/CFT correspondence. For supersymmetric backgrounds of the form $AdS_5 \times H$, we characterize via holomorphy a large family of supersymmetric $D3$-brane probes wrapped on H. We then quantize this classical family of probe solutions to obtain a BPS spectrum which describes $D3$-brane configurations on H. For the particular examples $H = T^{1,1}$ and $H = S^5$, we match the BPS spectrum to the spectrum of chiral baryonic operators in the dual gauge theory.

Keywords: AdS-CFT and dS-CFT Correspondence, D-branes.

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1. Introduction

The AdS/CFT correspondence [1] is a remarkable example of the long-suspected relationship between string theories and gauge theories. Yet, despite the many compelling pieces of evidence that exist for a duality between type-IIB string theory on $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ super Yang-Mills theory, we still do not understand such a fundamental issue as how the Yang-Mills theory encodes the full spectrum of the string theory.

One difficulty, of course, is that even the perturbative string spectrum in the AdS background is not known, owing to the presence of Ramond-Ramond five-form flux. Nevertheless, we can circumvent this difficulty in various ways. For instance, the massless modes of the string just give rise to the spectrum of type-IIB supergravity in the AdS background, which can be directly computed [2]. These supergravity states fall into short representations of the superconformal algebra and can be matched to chiral primary operators in the Yang-Mills theory.

Another more recent approach to this problem has been to take a Penrose limit in which the $AdS_5 \times S^5$ background reduces to a pp-wave background [3, 4] for which the classical string spectrum can be computed exactly in a Green–Schwarz formalism [5]. In this limit the perturbative string states can be matched to "nearly chiral" operators which have large charge under a U(1) subgroup of the global SU(4) $\mathcal{R}$-symmetry of the Yang-Mills theory [6].

In this paper, we approach the problem of how the Yang-Mills theory describes "stringy" objects by studying the non-perturbative sector of BPS states which arise from supersymmetric $D3$-branes in the AdS background. Paradoxically, despite the fact that we can't quantize a string in this background, our main result is to quantize a supersymmetric $D3$-brane, which is possible because the brane couples very simply to the Ramond-Ramond flux.

Supersymmetry plays an essential role in our work, so we consider a general background of the form $AdS_5 \times H$ which preserves at least eight supercharges. Here H is a compact Einstein five-fold of positive curvature which supports N units of five-form flux,

$$\int_H F_5 = N\,. \tag{1.1}$$

In this background we study supersymmetric $D3$-branes which wrap three-cycles in H and appear as particles in AdS_5.

Such backgrounds have already been studied extensively [7]–[9] in the context of the AdS/CFT correspondence. Nor is the idea of studying supersymmetric $D3$-branes wrapped on three-cycles in H new [10]–[12]. In these, by now classic, references, the authors study supersymmetric $D3$-branes wrapped on topologically non-trivial three-cycles in $H = \mathbb{RP}^5$, $T^{1,1}$, and $S^5/\mathbb{Z}_3$. In each of these cases, due to the symmetry of H, a supersymmetric $D3$-brane can wrap any of a family of three-manifolds $\Sigma \subset H$. When the corresponding collective coordinates of the brane are quantized, the resulting zero-modes can be matched directly to a set of pfaffian or di-baryonic chiral operators in the dual gauge theories.

In a similar vein, supersymmetric $D3$-branes which wrap topologically trivial three-cycles in H have also been studied. The best example of such a brane is the giant graviton [13] on $H = S^5$. It is a $D3$-brane which wraps a topologically trivial S^3 but is still supersymmetric [14, 15] due to the angular momentum it carries as it orbits about a transverse circle in H. Largely in analogy to the di-baryonic operators which describe non-trivially wrapped branes, the authors of [16] have proposed that the BPS states associated to giant gravitons correspond to sub-determinant operators in the $\mathcal{N} = 4$ Yang-Mills theory.

These examples suggest a general correspondence between the spectrum of BPS states arising from $D3$-branes wrapped on three-cycles in H and the spectrum of chiral baryonic operators in the dual gauge theory. However, as insightfully observed by Berenstein, Herzog, and Klebanov [17], we still have some things to learn before we fully understand this correspondence.

For instance, as we shall review in section 2, the $\mathcal{N} = 1$ gauge theory dual to the $H = T^{1,1}$ background possesses a tower of chiral baryonic operators of increasing conformal dimension Δ. The baryons at the base of the tower correspond to the zero-modes of a supersymmetric $D3$-brane. But what $D3$-brane states do the baryons of higher Δ describe? The question of what this baryon spectrum really "means" cuts to the heart of how the gauge theory describes the geometry of a brane configuration on H. Yet, as we show, this question is tractable, even simple, because of supersymmetry.[1]

[1]We are hopeful that our results in understanding the correspondence between BPS states of $D3$-branes

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The simplest guess would be that these baryons just describe multi-particle states consisting of some zero-mode of the $D3$-brane plus particles in the supergravity multiplet on H. That guess is generally wrong. Although some of the baryons of higher Δ factor into products of chiral operators consistent with this hypothesis, most baryons do not factor in any simple way [17]. So we must somehow interpret these baryons as corresponding to new irreducible, BPS states of $D3$-branes.

In practical terms, the crux of this problem is to identify the spectrum of BPS states in the string theory which arise from supersymmetric $D3$-branes wrapped on three-cycles in a given (possibly trivial) homology class of H. As it turns out, once we have computed the BPS spectrum, we can trivially match it to the spectrum of chiral baryons in examples such as $H = T^{1,1}$ or $H = S^5$ for which the dual gauge theory is known.

One approach to computing the above BPS spectrum is to use the Dirac-Born-Infeld action for a probe $D3$-brane. The DBI action is appropriate for describing a weakly-curved D-brane at weak string coupling. Hence this description of the $D3$-brane is appropriate in the supergravity regime of a weakly-curved background, corresponding to large 't Hooft coupling in the gauge theory. Provided the Einstein metric on H is known, one can directly use the DBI action to compute perturbatively the spectrum of small fluctuations about a particular supersymmetric $D3$-brane configuration in H.

The authors of [17] performed this computation for the well-known supersymmetric $D3$-branes in $H = T^{1,1}$ and, amidst a plethora of modes, they identified a subset of BPS-saturated states having quantum numbers consistent with some of the chiral baryonic operators of higher Δ. Hence they provided compelling evidence for the general idea that chiral baryonic operators of higher Δ should correspond to additional BPS states of wrapped $D3$-branes.

We provide a complementary analysis of the BPS spectrum. In any quantum mechanics problem, a thorough understanding of the space $\mathcal{M}$ of classical solutions is often an essential ingredient in quantization. So rather than starting from a particular supersymmetric $D3$-brane and studying its small fluctuations, we begin in section 3 by asking, "How can we characterize the general supersymmetric $D3$-brane configuration on H?" Luckily, Mikhailov [19] has already provided such a characterization via holomorphy, and we can apply his results directly to describe $\mathcal{M}$. In the example $H = T^{1,1}$, we explicitly present many new supersymmetric $D3$-brane configurations on H, all of which are generalizations of the giant graviton solution on S^5 to non-trivially wrapped branes on H.

In section 4, we quantize the classical, supersymmetric $D3$-brane configurations which we found in section 3. That is, we determine the Hilbert space of BPS states associated to $\mathcal{M}$ as well as the representations in which these states transform under the various global symmetries. This exercise is a direct generalization of the quantization of collective coordinates discussed in [10, 11], and [12]. We discuss the examples $H = T^{1,1}$ and $H = S^5$ in detail. As a result, we find an immediate correspondence between BPS states and chiral

and chiral operators of the gauge theory will be useful in understanding the analogous correspondence of non-BPS states and non-chiral operators, which in the pp-wave limit on S^5 has already been considered in [18].

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baryonic operators in the dual gauge theories. We finally discuss some issues relating to multi-brane states in string theory and factorizable baryonic operators in gauge theory.

2. $T^{1,1}$, the conifold, baryons, and all that

Although much of the discussion in sections 3 and 4 holds for a general H which preserves supersymmetry, the example $H = T^{1,1}$ is very good to keep in mind. In order that this paper be self-contained and to establish notation, we now review some aspects of the correspondence between type-IIB string theory on $AdS_5 \times T^{1,1}$ and its dual $\mathcal{N} = 1$ gauge theory [7]–[9]. We also review the relation, which we generalize in section 4, between wrapped $D3$-branes and baryonic operators proposed in [11]. We finally describe the observations of [17] which inspired our work. So in this section, $H = T^{1,1}$.

2.1 The geometry of $T^{1,1}$ and the conifold

Recall that H is the homogeneous space $\mathrm{SU}(2) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$, where the $\mathrm{U}(1)$ is a diagonal subgroup inside a maximal torus of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The manifold underlying $\mathrm{SU}(2)$ is S^3, and the S^1 fiber of the Hopf fibration $S^1 \to S^3 \to S^2$ corresponds to a maximal torus of $\mathrm{SU}(2)$. So the quotient $\mathrm{SU}(2) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$ can also be described as an S^1-fibration over $S^2 \times S^2$; equivalently H is a principal $\mathrm{U}(1)$-bundle over $S^2 \times S^2$.

This five-manifold H possesses a unique Einstein metric for which the $AdS_5 \times H$ background preserves eight supercharges. This Einstein metric is naturally written in coordinates adapted to the S^1-fibration. Let (θ_1, ϕ_1) and (θ_2, ϕ_2) be the usual angular coordinates on the two-spheres in the $S^2 \times S^2$ base, and let $\psi \in [0, 4\pi)$ be an angular coordinate on the S^1 fiber. Then the Einstein metric is

$$ds^2_H = \frac{1}{9}\left(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i\right)^2 + \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right), \tag{2.1}$$

which satisfies

$$R_{mn} = 4 g_{mn}\,. \tag{2.2}$$

An important property of this metric is that it possesses an $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ group of isometries. The $\mathrm{U}(1)$ isometry is generated by $\partial/\partial\psi$, and the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ isometries act on the $S^2 \times S^2$ base, which we can regard as the Kähler-Einstein manifold $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Topologically, H is equivalent to $S^2 \times S^3$, so that $\mathrm{H}_3(H) = \mathbb{Z}$. H has two obvious families of three-spheres, which arise as the loci of fixed (θ_1, ϕ_1) or fixed (θ_2, ϕ_2) on the $S^2 \times S^2$ base. We denote the corresponding classes in H_3 by A and B. Both A and B generate H_3, and since the members of the respective families are oppositely oriented, $A = -B$.

The superconformal theory dual to the $AdS_5 \times H$ background arises from the infrared limit of the worldvolume theory on coincident $D3$-branes at the conifold singularity X. Recall that X can be presented algebraically as the locus of points $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$ which satisfy

$$z_1 z_2 - z_3 z_4 = 0\,. \tag{2.3}$$

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This locus is preserved by the $\mathbb{C}^{\times}$-action

$$(z_1, z_2, z_3, z_4) \to (\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4)\,, \qquad \lambda \in \mathbb{C}^{\times}\,. \tag{2.4}$$

When λ is real, the $\mathbb{C}^{\times}$-action amounts to scaling in $\mathbb{C}^4$, so we see that X is a cone. This $\mathbb{C}^{\times}$-action is free away from the origin of $\mathbb{C}^4$, where X possesses an isolated conical singularity.

X is actually a Calabi-Yau cone, and a Ricci-flat Kähler metric on X can be determined very simply using symmetries. By a linear change of variables in $\mathbb{C}^4$, we can re-write the defining equation (2.3) of X as

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\,. \tag{2.5}$$

From (2.5), we see that $\mathrm{SO}(4)$ acts in a natural way on X, and we look for a Kähler metric with this group acting by isometries. The associated Kähler potential K can also be chosen to be invariant under $\mathrm{SO}(4)$ and hence must be a function only of

$$|z|^2 \equiv |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2\,. \tag{2.6}$$

Because $|z|^2$ is invariant under the $\mathrm{U}(1)$ part of the $\mathbb{C}^{\times}$-action in (2.4), under which

$$(z_1, z_2, z_3, z_4) \to (e^{i\alpha} z_1, e^{i\alpha} z_2, e^{i\alpha} z_3, e^{i\alpha} z_4)\,, \qquad \alpha \in [0, 2\pi)\,, \tag{2.7}$$

the Kähler metric actually has an $\mathrm{SO}(4) \times \mathrm{U}(1)$ group of isometries. As the holomorphic three-form Ω of X,

$$\Omega = \frac{dz_2 \wedge dz_3 \wedge dz_4}{z_1}\,, \tag{2.8}$$

has charge two under (2.7), we identify this $\mathrm{U}(1)$ isometry with an $\mathcal{R}$-symmetry.

If the Kähler metric is to be conical, K must transform homogeneously under the scaling with real λ in (2.4). Thus, $K = |z|^{2\gamma}$ for some exponent γ. To fix γ, we note that Ω transforms as $\Omega \to \lambda^2\, \Omega$ under (2.4). Under the same scaling, the Kähler form ω,

$$\omega = -i \frac{\partial^2 K}{\partial z_i\, \partial \overline{z}_j}\, dz_i {\wedge} d\overline{z}_j\,, \tag{2.9}$$

transforms as $\omega \to \lambda^{2\gamma}\, \omega$. The Calabi-Yau condition finally implies that

$$\omega{\wedge}\omega{\wedge}\omega = -i\, \Omega{\wedge}\overline{\Omega}\,, \tag{2.10}$$

which determines $\gamma = 2/3$.

Let us return to the defining equation (2.3) of X. We can solve this equation by introducing four homogeneous coordinates A_1, A_2, B_1, B_2, and setting

$$z_1 = A_1 B_1\,, \qquad z_2 = A_2 B_2\,, \qquad z_3 = A_1 B_2\,, \qquad z_4 = A_2 B_1\,. \tag{2.11}$$

The coordinates (A_1, A_2, B_1, B_2) are homogeneous (as opposed to true) coordinates on X because the affine coordinates (z_1, z_2, z_3, z_4) of $\mathbb{C}^4$ are left invariant under the $\mathbb{C}^{\times}$-action,

$$(A_1,\, A_2,\, B_1,\, B_2) \to (\mu A_1,\, \mu A_2,\, \mu^{-1} B_1,\, \mu^{-1} B_2)\,, \qquad \mu \in \mathbb{C}^{\times}\,. \tag{2.12}$$

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Away from the singularity at the origin, we can fix the real part of this action by requiring that

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 \,. \tag{2.13}$$

We are left with the U(1)-action given by

$$(A_1, A_2, B_1, B_2) \to (e^{i\beta} A_1, e^{i\beta} A_2, e^{-i\beta} B_1, e^{-i\beta} B_2)\,, \qquad \beta \in [0, 2\pi)\,. \tag{2.14}$$

In terms of the homogeneous coordinates, X is thus obtained by restricting to the locus given by (2.13) and dividing by the residual U(1)-action of (2.14).

We remark that the introduction of homogeneous coordinates on X is much less ad hoc than it may appear. X is a toric variety [20], and toric varieties are generalizations of projective and weighted projective spaces. As such, any toric variety is endowed with a set of global homogeneous coordinates [21] which function much like the usual projective coordinates of projective space.

The five-manifold H can be obtained from X by further restricting to the locus

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 = 1\,. \tag{2.15}$$

This restriction amounts to a quotient by the scaling part of the $\mathbb{C}^\times$-action (2.4) on X, after the fixed point at the origin is excised. To see that the description of H using the homogeneous coordinates agrees with our earlier description of H as $\mathrm{SU}(2) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$, note that the locus determined by (2.15) is $S^3 \times S^3 = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and the residual U(1)-action (2.14) provides the correct quotient.

After removing the singularity at the origin, X is topologically $\mathbb{R}_+ \times H$. The cone metric on X,

$$ds_X^2 = dr^2 + r^2 ds_H^2\,, \tag{2.16}$$

is automatically Ricci-flat due to the Einstein relation (2.2). This metric has the same $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1) = \mathrm{SO}(4)\times\mathrm{U}(1)$ group of isometries as the Calabi-Yau metric discussed earlier, and it coincides (up to normalization) with that metric. The only question is how r is related to $|z|$. Since the metric is homogeneous in r of degree 2 and homogeneous in $|z|$ of degree 4/3, we must have

$$r = |z|^{2/3} = \left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2\right)^{1/3}\,. \tag{2.17}$$

We have discussed the geometry of $T^{1,1}$ (and the conifold) in some detail because this example nicely illustrates the general properties of any H which preserves supersymmetry. We will use these properties in section 3 when we discuss supersymmetric $D3$-branes.

2.2 The conifold gauge theory

Having introduced the conifold X, we now recall the dual superconformal theory which arises from the infrared limit of the worldvolume theory on coincident $D3$-branes at the conical singularity of X. This theory can be described as an infrared fixed-point of an $\mathcal{N} = 1$ supersymmetric, $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory. The gauge theory includes four $\mathcal{N} = 1$ chiral matter multiplets A_1, A_2, B_1, and B_2. The fields A_1 and A_2 transform in

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the $(\mathbf{N},\overline{\mathbf{N}})$ representation of the gauge group, and the fields B_1 and B_2 transform in the $(\overline{\mathbf{N}},\mathbf{N})$ representation. The chiral matter multiplets are coupled by a superpotential

$$W = \frac{\lambda}{2}\left(\mathrm{tr}\,[A_1 B_1 A_2 B_2] - \mathrm{tr}\,[A_1 B_2 A_2 B_1]\right), \tag{2.18}$$

where λ is a (dimensionful) coupling constant.

This gauge theory possesses an anomaly-free U(1) $\mathcal{R}$-symmetry under which the lowest, scalar components of A_1, A_2, B_1, and B_2 have charge $R = 1/2$. Hence the corresponding conformal dimension of these fields at the infrared fixed-point is $\Delta = (3/2)\,R = 3/4$, and the superpotential W is a marginal deformation of the pure gauge theory at this point.

The $\mathcal{R}$-symmetry of the gauge theory corresponds to the U(1) isometry arising from rotations in the S^1 fiber of H. Note that (2.7) and (2.11), together with the $\mathbb{Z}_2$ symmetry which exchanges the S^2 factors in the base of H, imply that the homogeneous coordinates of X also have charge $R = 1/2$. Further, under a radial scaling on X, the relation (2.17) implies that

$$r\frac{\partial}{\partial r} z_i = \frac{3}{2} z_i\,, \qquad i = 1,\ldots,4\,, \tag{2.19}$$

so the conformal dimension of the homogeneous coordinates is similarly $\Delta = 3/4$.

The $\mathrm{SU}(2)\times\mathrm{SU}(2)$ isometry of H also appears as a global $\mathrm{SU}(2)\times\mathrm{SU}(2)$ flavor symmetry in the gauge theory. Under this symmetry, the chiral fields (and the corresponding homogeneous coordinates) transform as $(\mathbf{2},\mathbf{1})$ for (A_1, A_2) and $(\mathbf{1},\mathbf{2})$ for (B_1, B_2).

However, the gauge theory in addition possesses a non-anomalous global $\mathrm{U}(1)_{\mathcal{B}}$ baryon-number symmetry which does not arise from an isometry of H. Under this $\mathrm{U}(1)_{\mathcal{B}}$ symmetry, A_1 and A_2 have charge $+1$, and B_1 and B_2 have charge -1. Not coincidentally, these charges correspond to the charges in (2.14) of the homogeneous coordinates under the U(1) subgroup of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ by which we quotient to obtain $T^{1,1}$.

To understand better the origin of the baryon-number symmetry, we first note that, in analogy to the six real scalars $\phi_1,\ldots,\phi_6$ of the $\mathcal{N}=4$ Yang-Mills theory which parametrize the positions of the $D3$-branes in $\mathbb{R}^6$, the complex scalars in the chiral multiplets A_1, A_2, B_1, B_2 should parametrize the positions of the $D3$-branes on X. To show this, we observe that the F-term equations which follow from (2.18) imply that, modulo descendants,

$$B_1 A_i B_2 = B_2 A_i B_1\,, \qquad A_1 B_j A_2 = A_2 B_j A_1\,, \qquad i,j = 1,2\,. \tag{2.20}$$

Consequently, if we introduce the chiral primary adjoints Z_1, Z_2, Z_3, and Z_4 defined by

$$Z_1 = A_1 B_1\,, \qquad Z_2 = A_2 B_2\,, \qquad Z_3 = A_1 B_2\,, \qquad Z_4 = A_2 B_1\,, \tag{2.21}$$

then these operators commute modulo descendants, and their N eigenvalues clearly satisfy the equation (2.3) defining X. Thus, as we already suspect from the action of the global symmetries, the chiral fields A_1, A_2, B_1, and B_2 are matrix generalizations of the homogeneous coordinates on X. The gauge-invariant mesonic operators made from single-traces of products of Z_1, Z_2, Z_3, and Z_4 then describe the locations of the $D3$-branes on X, and all of these operators are uncharged under $\mathrm{U}(1)_{\mathcal{B}}$.

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On the other hand, because the gauge group is $\mathrm{SU}(N)\times\mathrm{SU}(N)$ with chiral matter in bi-fundamental representations, the gauge theory admits baryonic operators made from anti-symmetrizing over the gauge indices of any N bi-fundamental chiral fields. For instance, we have chiral primary operators

$$\begin{aligned}\mathcal{B}^{(A)}_{i_1\cdots i_N} &= \frac{1}{N!}\,\epsilon_{\alpha_1\cdots\alpha_N}\epsilon^{\beta_1\cdots\beta_N}\prod_{n=1}^{N}(A_{i_n})^{\alpha_n}_{\beta_n}\ ,\\ \mathcal{B}^{(B)}_{j_1\cdots j_N} &= \frac{1}{N!}\,\epsilon^{\alpha_1\cdots\alpha_N}\epsilon_{\beta_1\cdots\beta_N}\prod_{n=1}^{N}(B_{j_n})^{\beta_n}_{\alpha_n}\ ,\end{aligned} \tag{2.22}$$

where the α, β indices are gauge indices of $\mathrm{SU}(N)\times S\,\mathrm{U}(N)$ and the i, j indices are flavor indices of $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Note that because we anti-symmetrize over gauge-indices, $\mathcal{B}^{(A)}_{i_1\cdots i_N}$ and $\mathcal{B}^{(B)}_{j_1\cdots j_N}$ are automatically symmetric in the flavor indices. Hence these operators transform under the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ flavor symmetry as $(\mathbf{N+1},\mathbf{1})$ and $(\mathbf{1},\mathbf{N+1})$. They also have respective charges $\pm N$ under $\mathrm{U}(1)_{\mathcal{B}}$, justifying the identification of this symmetry with baryon-number.

We now introduce a schematic notation for baryonic operators, suppressing gauge indices and denoting

$$\mathcal{B}^{(A)}_{i_1\cdots i_N}\equiv\epsilon_1\epsilon^2(A_{i_1},\ldots,A_{i_N})\,,\qquad \mathcal{B}^{(B)}_{j_1\cdots j_N}\equiv\epsilon^1\epsilon_2(B_{j_1},\ldots,B_{j_N})\,. \tag{2.23}$$

Here $\epsilon_1\equiv\epsilon_{\alpha_1\cdots\alpha_N}$ and $\epsilon_2\equiv\epsilon_{\beta_1\cdots\beta_N}$ are abbreviations for the completely anti-symmetric tensors for the respective $\mathrm{SU}(N)$ factors of the gauge group.

The baryonic operators $\mathcal{B}^{(A)}$ and $\mathcal{B}^{(B)}$ in (2.23) both have $\mathcal{R}$-charge $N/2$ and conformal dimension $\Delta=(3/4)N$. This fact makes them candidates to describe wrapped $D3$-branes. For if R denotes the curvature scale of the supergravity background,

$$R^4\propto g_s\alpha'^2N\,, \tag{2.24}$$

then the mass m of a $D3$-brane wrapped on H scales as

$$m\propto\frac{R^3}{g_s\alpha'^2}\,. \tag{2.25}$$

So operators in the gauge theory which correspond to the BPS states arising from a supersymmetric $D3$-brane wrapped on H must be chiral primary of dimension

$$\Delta\propto mR\propto N\,. \tag{2.26}$$

As first observed in [11], the baryons $\mathcal{B}^{(A)}$ and $\mathcal{B}^{(B)}$ indeed arise from $D3$-branes wrapped on three-spheres in the A and B families described in section 2.1. Three-spheres in both of these families minimize volume within their respective homology classes, and a brane wrapped about any of these three-spheres is supersymmetric. In order to determine what BPS states arise from such a wrapped brane, we must quantize the bosonic collective coordinates (θ_1,ϕ_1) or (θ_2,ϕ_2) associated to the location of the brane on the transverse S^2. The supersymmetry generators broken by the brane then create the additional states necessary to fill out the chiral multiplet.

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The SU(2) symmetry dictates that no potential is associated to the location of the $D3$-brane on the transverse S^2, but the brane does couple magnetically, via the Wess-Zumino term in the DBI action, to the N units of five-form flux on H. (See [17] for a very explicit demonstration of this statement.) So the quantization of the collective coordinates (θ_1, ϕ_1) or (θ_2, ϕ_2) reduces to the quantization of a charged particle moving on the complex plane in a perpendicular magnetic field of N flux quanta. Such a particle has $N+1$ degenerate ground-states, transforming in the $(N+1)$-dimensional representation of SU(2).

We naturally identify these $N+1$ BPS states arising from either the A or B cycles as corresponding to the respective baryonic operators $\mathcal{B}^{(A)}_{i_1 \cdots i_N}$ and $\mathcal{B}^{(B)}_{j_1 \cdots j_N}$. The SU(2) $\times$ SU(2) flavor quantum numbers are certainly consistent with this identification. Also, the baryon dimension $\Delta = (3/4)N$ can be checked against the volume of the wrapped three-spheres [11, 17].

Finally, this example illustrates that the $\mathrm{U}(1)_{\mathcal{B}}$ baryon-number of the gauge theory represents a topological charge from the perspective of the string theory, as it measures the class of the corresponding wrapped $D3$-brane in H_3.

2.3 An abundance of baryons

Now, as observed by the authors of [17], we seem to have exhausted the obvious BPS states arising from supersymmetric $D3$-branes wrapped on three-cycles in H. However, we have certainly not exhausted all of the chiral baryonic operators in the conifold gauge theory.

Besides the chiral fields A_1 and A_2, we have composite chiral operators

$$A_{I;J} \equiv A_{i_1 \cdots i_{m+1}; j_1 \cdots j_m} \equiv A_{i_1} B_{j_1} \cdots A_{i_m} B_{j_m} A_{i_{m+1}} \,, \tag{2.27}$$

which also transform as $(\mathbf{N}, \overline{\mathbf{N}})$ under the gauge group and have charge one under $\mathrm{U}(1)_{\mathcal{B}}$. Here (I, J) denotes the SU(2) $\times$ SU(2) flavor representation of $A_{I;J}$. The F-term equations (2.20) again imply that, modulo descendants, the operator $A_{i_1 \cdots i_{m+1}; j_1 \cdots j_m}$ is fully symmetric on the flavor indices $\{i_1, \ldots, i_{m+1}\}$ and $\{j_1, \ldots, j_m\}$. Thus, for the operator in (2.27), $(I, J) = (\mathbf{m+2}, \mathbf{m+1})$.

Using any N operators of the general form in (2.27), we can form additional chiral primary baryons $\mathcal{B}^{(A)}_{R;S}$,

$$\mathcal{B}^{(A)}_{R;S} = t^{I_1 \cdots I_N; J_1 \cdots J_N}_{R;S} \, \epsilon_1 \epsilon^2 \left(A_{I_1;J_1} \,, \, \ldots \,, \, A_{I_N;J_N} \right) . \tag{2.28}$$

Here $t^{I_1 \cdots I_N; J_1 \cdots J_N}_{R;S}$ is a Clebsch-Gordan tensor coupling the product of flavor representations $(I_1, J_1) \otimes \cdots \otimes (I_N, J_N)$ to the irreducible flavor representation (R, S) of the baryon $\mathcal{B}^{(A)}_{R;S}$. We caution that a given flavor representation (R, S) generically appears with multiplicity in this product, so the notation $\mathcal{B}^{(A)}_{R;S}$ is only schematic and usually insufficient to specify a given baryon.

The more general baryons $\mathcal{B}^{(A)}_{R;S}$ in (2.28) transform in the same baryon-number sector as the baryon $\mathcal{B}^{(A)}_{i_1 \cdots i_N}$. Because baryon-number is identified as a topological charge measured by H_3 in the string theory, the BPS string states corresponding to the baryons $\mathcal{B}^{(A)}_{R;S}$ must arise from supersymmetric $D3$-branes still wrapped in the class of the A cycle on H.

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On the other hand, the new baryonic operators have dimensions Δ strictly greater than $(3/4)N$. Since the baryons with $\Delta = (3/4)N$ describe $D3$-branes wrapping the minimal-volume cycles in the A class, a natural interpretation of the baryons of higher dimension is that they represent BPS excitations of these branes.

One might hope that anyway a baryon of higher dimension could always be factored as a product of the minimal baryon $\mathcal{B}^{(A)}_{i_1\cdots i_N}$ and some chiral operator in the trivial baryon-number sector. A natural interpretation of such reducible baryons would be as multi-particle states representing the minimally-wrapped $D3$-brane and a topologically trivial, gravitonic excitation on H.

Indeed, some baryons clearly factor in this way — but others do not. A baryon $\mathcal{B}^{(A)}_{R;S}$ of the form (2.28) factors precisely when R is the representation which is fully-symmetric on the A flavor indices. In this case, the tensor relation

$$\epsilon_1\epsilon^1 \equiv \epsilon_{\alpha_1\cdots\alpha_N}\,\epsilon^{\gamma_1\cdots\gamma_N} = \delta^{\gamma_1}_{[\alpha_1}\cdots\delta^{\gamma_N}_{\alpha_N]} \tag{2.29}$$

implies that $\mathcal{B}^{(A)}_{R;S}$ can be factored as

$$\begin{aligned}\mathcal{B}^{(A)}_{R;S} &= t_{R;S}^{I_1\cdots I_N;J_1\cdots J_N}\ \epsilon_1\epsilon^1\left(O_{K_1;J_1}\,,\ \ldots\,,\ O_{K_N;J_N}\right)\,\cdot\,\epsilon_1\epsilon^2\left(A_{i_1}\,,\ \ldots\,,\ A_{i_n}\right)\,,\\ &= t_{R;S}^{I_1\cdots I_N;J_1\cdots J_N}\ \mathcal{B}^{(0)}_{K_1\cdots K_N;J_1\cdots J_N}\,\cdot\,\mathcal{B}^{(A)}_{i_1\cdots i_N}\,.\end{aligned} \tag{2.30}$$

Here each $O_{K;J}$ is an adjoint chiral operator having the general form

$$O_{K;J} \equiv O_{k_1\cdots k_m;j_1\cdots j_m} \equiv A_{k_1}B_{j_1}\cdots A_{k_m}B_{j_m}\,, \tag{2.31}$$

again fully-symmetric on flavor indices and transforming as $(\mathbf{m+1},\mathbf{m+1})$ under the flavor symmetry. As indicated above,

$$\mathcal{B}^{(0)}_{K_1\cdots K_N;J_1\cdots J_N} \equiv \epsilon_1\epsilon^1\left(O_{K_1;J_1}\,,\ \ldots\,,\ O_{K_N;J_N}\right) \tag{2.32}$$

is a chiral primary "baryonic" operator in the trivial sector of $\mathrm{U}(1)_{\mathcal{B}}$ made by anti-symmetrizing the gauge indices of N chiral operators of the form (2.31). Also, in (2.30) we implicitly symmetrize over the indices in $K_1\otimes i_1,\ldots,K_N\otimes i_N$ to couple these representations respectively to $I_1,\ldots,I_N$.

For the case $m=0$ in (2.31), we include the constant tensor δ^γ_α in the general class of operators O. Hence the baryons of the form $\mathcal{B}^{(0)}$ include the sub-determinants,

$$\epsilon_{\alpha_1\cdots\alpha_n\alpha_{n+1}\cdots\alpha_N}\,\epsilon^{\gamma_1\cdots\gamma_n\alpha_{n+1}\cdots\alpha_N}\prod_{k=1}^{n}\left(O_{I_k;J_k}\right)^{\alpha_k}_{\gamma_k}\,,\qquad 0<n<N\,, \tag{2.33}$$

as well as the trivial operator $\mathbf{1}$ corresponding to $n=0$ in (2.33). These baryons $\mathcal{B}^{(0)}$ of dimension $\Delta\sim N$, generally identified as giant gravitons [16], correspond to BPS states of $D3$-branes wrapped on topologically trivial three-cycles in H, and the baryons of dimension $\Delta\sim 1$ correspond to multi-particle supergravity excitations of the metric and the four-form potential C_4 on H.

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For instance, the authors of [17] considered a simple class of baryons of the form

$$\mathcal{B}^{(A)}_{R;S} = t_R^{I_1 i_2 \cdots i_N}\ \epsilon_1 \epsilon^2 \left(A_{I_1;S}\,,\, A_{i_2}\,,\, \ldots\,,\, A_{i_N} \right). \tag{2.34}$$

If R is the fully-symmetric representation, then $\mathcal{B}^{(A)}_{R;S}$ factors as

$$\begin{aligned} \mathcal{B}^{(A)}_{R;S} &= t_R^{I_1 \cdots i_N}\, \epsilon_1 \epsilon^1 \left(O_{K_1;S}\,,\, \mathbf{1}\,,\, \ldots\,,\, \mathbf{1} \right) \cdot \epsilon_1 \epsilon^2 \left(A_{i_1}\,,\, A_{i_2}\,,\, \ldots\,,\, A_{i_N} \right)\,, \\ &= t_R^{I_1 \cdots i_N}\ \operatorname{tr}\left[O_{K_1;S} \right] \cdot \mathcal{B}^{(A)}_{i_1,\ldots,i_N}\,. \end{aligned} \tag{2.35}$$

For the other representations R, the baryons in (2.34) do not factor and represent "new" BPS states which must be considered. These baryons appear in a tower of evenly-spaced $\mathcal{R}$-charges $N/2$, $(N/2)+1$, $(N/2)+2$, $\ldots$, with corresponding representations under the SU(2) B flavor symmetry being $S = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots$.

In [17], the authors then perturbatively computed the spectrum of small fluctuations of a minimally-wrapped $D3$-brane in the A class on H and found a set of BPS fluctuations also having these quantum numbers. They proposed that these fluctuations should correspond to the additional baryons in (2.34). But to study the general baryon in (2.28), we will apply a slightly different analysis.

3. Classical supersymmetric $D3$-branes

The additional chiral baryons suggest additional BPS $D3$-brane states. To account for these states, we need to find new classical, supersymmetric $D3$-brane configurations on H. So what is the general description of a supersymmetric $D3$-brane which wraps a three-cycle in H?

If we consider compactifying the type-IIB theory on a Calabi-Yau three-fold X, the analogous question has a simple answer: supersymmetric D-brane configurations are characterized by holomorphy in X [22, 23]. In the Freund-Rubin compactification on H, might holomorphy also play a role?

Mikhailov [19] has answered this question affirmatively, using the fact that compactification on H is closely related to compactification on an associated Calabi-Yau cone X. As we shall discuss, supersymmetric $D3$-brane configurations in H arise from holomorphic surfaces in X.

First, we mention a caveat to our analysis. In characterizing supersymmetric $D3$-brane configurations on H, for simplicity we will ignore the degrees of freedom associated to the worldvolume fermions and gauge field on the brane. In particular, we assume that the U(1) line bundle supported by the brane has a flat, topologically trivial connection. The latter assumption is too strong in general. Whenever the $D3$-brane wraps a three-manifold in H which has a non-trivial fundamental group π_1, we should also look for supersymmetric configurations with Wilson or 't Hooft loops turned on in the worldvolume. Examples of these configurations occur for branes wrapped on $S^3/\mathbb{Z}_3$ in the lens space $H = S^5/\mathbb{Z}_3$, discussed thoroughly in [12].

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3.1 Supersymmetric branes from holomophic surfaces

When H preserves supersymmetry, there is always a unique Calabi-Yau cone X associated to H [7]–[9]. (See also [34] and [25] for a nice mathematical discussion of these facts.) Just as for $T^{1,1}$, the presence of a Killing spinor on H implies that the Einstein metric of H satisfies $R_{mn} = 4\,g_{mn}$. Thus, if X is the real cone over H, the Einstein condition implies that the cone metric of X, $ds^2_X = dr^2 + r^2\,ds^2_H$, is Ricci-flat, and the Killing spinor on H lifts to a covariantly-constant spinor on X.

As we know well from the AdS/CFT correspondence, the type-IIB compactification on $M_4 \times X$, where M_4 denotes flat Minkowski space, is closely related to the type-IIB compactification on $AdS_5 \times H$. Namely, if N coincident $D3$-branes fill M_4 and sit at $r = 0$ in X, then the branes warp the product metric on $M_4 \times X$ and source N units of five-form flux through H. Taking the near-horizon limit of the corresponding supergravity solution produces the $AdS_5 \times H$ background.

With these observations, we can easily motivate Mikhailov's description of supersymmetric $D3$-branes in H. Consider the euclidean theory on $\mathbb{R}^4 \times X$, with N $D3$-branes filling $\mathbb{R}^4$ and one $D3$-brane wrapped on a holomorphic surface S in X. We assume that S intersects H, embedded at $r = 1$ in X, in some three-manifold Σ.

This brane configuration is supersymmetric. Just as above, we consider the supergravity background produced by the N $D3$-branes and take the near-horizon limit. In this limit, the radial direction of X becomes a geodesic γ in euclidean AdS_5, and the probe brane wrapped upon S in X becomes a probe brane wrapped on a four-manifold in $\gamma \times H$. The $\mathrm{SO}(5,1)$ global symmetry of euclidean AdS_5 can be used to rotate γ into any other geodesic in this space, and so we can assume that γ becomes a time-like geodesic upon Wick-rotation back to Minkowski signature in AdS_5.

This procedure produces a supersymmetric $D3$-brane wrapped on H, and, running the procedure backwards, any supersymmetric $D3$-brane wrapped on H can be lifted to a holomorphic surface S in X. So we can identify the space $\mathcal{M}$ which parametrizes classical, supersymmetric $D3$-brane configurations as equivalently parametrizing holomorphic surfaces in X (which intersect the locus at $r = 1$ transversely).

Because X possesses many holomorphic surfaces, we have found the additional supersymmetric $D3$-brane configurations which we need to get the full BPS spectrum. Yet, morally speaking, all of these configurations are a simple generalization of the giant graviton solution [13] for $H = S^5$, in that they all describe branes which carry angular momentum around a circular orbit in H.

To be more explicit, we recall that the Einstein metric on H always possesses a $\mathrm{U}(1)$ isometry, generated by a Killing vector $\partial/\partial\phi$, which corresponds to the $\mathcal{R}$-symmetry of the $\mathcal{N} = 1$ superconformal algebra. Provided that this $\mathrm{U}(1)$-action is regular, meaning that its orbits are compact and of uniform length, as we now assume, then H is a principal $\mathrm{U}(1)$-bundle over a positively-curved Kähler-Einstein manifold[2] V.

Let τ be a unit-speed parameter along the geodesic γ in AdS_5, and let ϕ be a unit-speed parameter along the $\mathrm{U}(1)$ fiber over V. At the instant $\tau = 0$, corresponding to $r = 1$

[2]If these orbits are only compact, then V is an orbifold [26].

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in X, the $D3$-brane configuration on H is described by the three-manifold Σ. Holomorphy of S in X then implies that the $D3$-brane configuration in $\gamma \times H$ depends only on τ and ϕ through the combination $\tau + \phi$. As a result, the brane is described just by translating Σ at unit-speed around the U(1) fiber over V.

One great advantage of our characterization of supersymmetric $D3$-brane configurations via holomorpy is that it does not require explicit knowledge of the Einstein metric on H. We must only know the appropriate complex structure on X, which is usually obvious. Also, at least in the examples $H = T^{1,1}$ and $H = S^5$, we can very simply parametrize the relevant holomorphic surfaces in X, and thus describe $\mathcal{M}$ directly.

3.2 Holomorphic surfaces in the conifold

Let $H = T^{1,1}$ and X be the conifold. Holomorphic surfaces S in X are easy to describe. Because we defined X through an embedding in $\mathbb{C}^4$, a very obvious set of holomorphic surfaces arise from the intersection of hypersurfaces in $\mathbb{C}^4$ with X. The general hypersurface is simply determined as the vanishing locus of a polynomial P in the affine coordinates (z_1, z_2, z_3, z_4) of $\mathbb{C}^4$.

However, the naive approach of using hypersurfaces in $\mathbb{C}^4$ to describe holomorphic surfaces in X will not quite work. Some polynomials, for instance $P = z_1 z_2 - z_3 z_4$, restrict to zero on X and therefore do not actually determine a holomorphic surface in X. Furthermore, if we were to attempt to parametrize surfaces in X by parametrizing polynomials in the affine coordinates of $\mathbb{C}^4$, we would have to face the issue that, on X, these polynomials are only well-defined up to the addition of the polynomials which vanish identically on X. Finally, any holomorphic surface in X which does arise from a hypersurface in $\mathbb{C}^4$ describes a $D3$-brane which is wrapped in the trivial homology class of H, and we are most interested in branes wrapped in the (non-trivial) A class of H.

These difficulties can all be overcome if we use the homogeneous coordinates A_1, A_2, B_1, and B_2 on X. If P is a polynomial in the these coordinates which transforms homogeneously under the U(1)-action in (2.14), then the vanishing locus of P is a well-defined holomorphic surface in X. Unlike the affine coordinates of $\mathbb{C}^4$, though, there are no algebraic relations among the homogeneous coordinates on X.

In addition, polynomials P which are charged under the U(1)-action in (2.14) can be used to describe topologically non-trivial wrappings of branes on H. In fact, the charge of P under this U(1) corresponds to the class of the wrapped three-cycle in $\mathrm{H}_3(H) = \mathbb{Z}$. For instance, if P has charge zero, then the relations (2.21) imply that P arises from the restriction of a hypersurface in $\mathbb{C}^4$. As we have mentioned, these hypersurfaces all describe trivially wrapped $D3$-branes on H.

To describe $D3$-branes on H which wrap in the A class of H, we consider polynomials $P(A_1, A_2, B_1, B_2)$ which have charge one. Such a polynomial has the form

$$\begin{aligned} P(A_1, A_2, B_1, B_2) = {} & c_1 A_1 + c_2 A_2 + c_{11;1} A_1^2 B_1 + c_{12;1} A_1 A_2 B_1 + c_{22;1} A_2^2 B_1 + \\ & + c_{11;2} A_1^2 B_2 + c_{12;2} A_1 A_2 B_2 + c_{22;2}\, A_2^2 B_2 + \\ & + (\text{terms of higher degree in } A_1, A_2)\,. \end{aligned} \tag{3.1}$$

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Here the c's are just arbitrary complex coefficients for each monomial in P. To write this expression more concisely, we introduce multi-indices I and J so that

$$P(A_1, A_2, B_1, B_2) = \sum_{I,J} c_{I;J}\, A^I B^J \,. \tag{3.2}$$

We can consider the coefficients $c_{I;J}$ as complex coordinates $(c_1,\, c_2,\, c_{11;1},\, \ldots)$ on the infinite-dimensional, complex vector space of polynomials P which have charge one. Each point in this vector space, other than the origin, determines a holomorphic surface in X. And, modulo the trivial fact that P and λP, for any $\lambda \in \mathbb{C}^\times$, vanish on the same locus in X, the points in this vector space describe distinct surfaces. Thus, we identify the classical configuration space $\mathcal{M}$ of supersymmetric branes wrapped in the A class on H as $\mathbb{CP}^\infty$, with homogeneous coordinates $[c_1 : c_2 : c_{11;1} : \ldots]$.

We can fairly easily visualize what branes some of these polynomials describe. As an example, let us consider a linear polynomial

$$P = c_1 A_1 + c_2 A_2 \,. \tag{3.3}$$

Both terms in P have $\mathcal{R}$-charge $1/2$, so the zero locus of P is fixed under the $\mathrm{U}(1)$ isometry of H, meaning that the brane described by P wraps a fixed three-manifold Σ independent of time. Since A_1 and A_2 descend to homogeneous coordinates on one $\mathbb{CP}^1$ factor of the $\mathbb{CP}^1 \times \mathbb{CP}^1$ base of H, P vanishes over a point, parametrized by c_1 and c_2, on this $\mathbb{CP}^1$. So this $D3$-brane configuration corresponds to one of the well-known supersymmetric $D3$-branes which wrap a minimal-volume S^3 in the A class.

One the other hand, when P is of the general form in (3.1), the various terms have different $\mathcal{R}$-charges and P transforms inhomogeneously under the $\mathcal{R}$-symmetry. Hence the general $D3$-brane configuration corresponds to some Σ orbiting the $\mathrm{U}(1)$ fiber over $\mathbb{CP}^1 \times \mathbb{CP}^1$. To obtain a more explicit description of Σ, we would actually have to solve the equation $P = 0$ on H — but of course the important issue is not so much the particular description of a given brane configuration, but the global description of the entire family $\mathcal{M}$ of configurations.

We can equally well consider $D3$-branes wrapped in other homology classes, corresponding to polynomials of other charges, and the above comments remain unaltered. $\mathcal{M}$ is always an infinite-dimensional projective-space whose homogeneous coordinates correspond to the coefficients in the homogeneous polynomials of fixed charge — except for a subtlety relating to the trivially-wrapped $D3$-branes. We will explain this point as we consider the example $H = S^5$, corresponding to $X = \mathbb{C}^3$.

3.3 Holomorphic surfaces in $\mathbb{C}^3$

Let (z_1, z_2, z_3) be affine coordinates on $X = \mathbb{C}^3$. The relevant $\mathrm{U}(1)$ isometry of $H = S^5$ corresponds to

$$(z_1, z_2, z_3) \rightarrow (e^{i\alpha}\, z_1, e^{i\alpha}\, z_2, e^{i\alpha}\, z_3)\,, \qquad \alpha \in [0, 2\pi)\,. \tag{3.4}$$

The space of orbits for this $\mathrm{U}(1)$-action on H is just $\mathbb{CP}^2$.

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Any holomorphic surface in X can be presented as a hypersurface determined by the zero locus of a polynomial $P(z_1, z_2, z_3)$. For concreteness, we parametrize the polynomial P as

$$\begin{aligned} P &= c + c_i\, z_i + c_{ij}\, z_i z_j + c_{ijk}\, z_i z_j z_k + \cdots \,, \\ &= \sum_I c_I\, z^I \,, \end{aligned} \tag{3.5}$$

where c, c_i, c_{ij}, c_{ijk}, $\ldots$, are complex coefficients and in the second line of (3.5) we introduce a multi-index I for brevity. As for the conifold, we regard these coefficients as homogeneous coordinates $[c : c_i : c_{ij} : c_{ijk} : \ldots]$ on a $\mathbb{CP}^\infty$ parameter space.

However, the new ingredient in this example is the presence of a constant term c in P. The associated point $[1 : 0 : 0 : 0 : \ldots]$ in $\mathbb{CP}^\infty$ corresponds to the nowhere-vanishing, constant polynomial $P = 1$. The constant polynomial of course does not correspond to any hypersurface in X nor a $D3$-brane configuration on H.

Further, the general construction of a supersymmetric $D3$-brane configuration on H requires that the holomorphic surface S determined by P intersects H transversely at $r = 1$. So points near $[1 : 0 : 0 : 0 : \ldots]$, for which $c \gg c_i, c_{ij}, c_{ijk}, \ldots$, do not determine surfaces which intersect H and thus do not determine any $D3$-brane configurations either.

In this case, the classical configuration space $\mathcal{M}$ does not consist of the full $\mathbb{CP}^\infty$ parameter space, but only an open subset of this space away from the bad point at $[1 : 0 : 0 : 0 : \ldots]$. Nevertheless, we will see that quantum mechanically this example is much like the simpler case for which $\mathcal{M} = \mathbb{CP}^\infty$.

Yet the fact that we include the constant term in the general polynomial is important. A simple example to consider is an affine polynomial of the form $P = z_1 - c$. For $|c| < 1$, this polynomial defines a hyperplane in X which intersects H in an S^3. As long as $c \neq 0$, the U(1)-action in (3.4) translates this S^3 around a circle in the H. So the affine polynomials describe the giant gravitons, and c controls the size of the wrapped S^3.

Another simple case to consider is a polynomial of arbitrary degree n but which depends on only a single coordinate such as z_1. In this case, we can factor the polynomial as

$$P = \prod_{i=1}^{n} (z_1 - e_i)\,. \tag{3.6}$$

Provided that all of the roots e_i satisfy $|e_i| < 1$, this polynomial represents a configuration of n giant gravitons of various sizes on H.

This example is important, as it illustrates the general principle that reducible surfaces, corresponding to factorizable polynomials, represent configurations of multiple branes. So when we quantize the space $\mathcal{M}$, which includes both irreducible and reducible surfaces, we automatically "second quantize" the $D3$-brane. We shall discuss some additional aspects of the relation of reducible surfaces to multi-brane states at the end of section 4.

3.4 A technical aside

The following is a somewhat technical aside, not necessary for reading the rest of the paper but possibly useful if one wishes to generalize the analysis which we performed for non-trivial wrappings in $H = T^{1,1}$ to other examples. In writing global expressions for

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the holomorphic surfaces in the conifold X, we naturally used the global homogeneous coordinates (A_1, A_2, B_1, B_2). The existence of such homogeneous coordinates is related to the fact that X is a toric variety. But for the general X, global homogeneous coordinates need not exist. Nevertheless, we can still very easily describe the classical configuration space $\mathcal{M}$.

To start, we will present a slightly different characterization of supersymmetric $D3$-branes which wrap three-cycles in H. H can be described as a principal U(1)-bundle over a Kähler-Einstein manifold V. This bundle is, in a suitable sense, holomorphic. If A is the unitary connection on this bundle arising from the metric of H, so that

$$ds_H^2 = (d\phi + iA)^2 + ds_V^2\,, \tag{3.7}$$

then the curvature $F = dA$ is proportional to the Kähler form ω of V. In particular, F is of type $(1,1)$ with respect to the complex structure of V. Hence, this U(1) bundle over V can always be considered to arise as the unit-circle bundle of some holomorphic line bundle over V.

A $D3$-brane configuration on H is described by a four-manifold in $\gamma \times H$. By the above remarks, we can regard $\gamma \times H$ (in euclidean signature) as the total space E of a holomorphic $\mathbb{C}^\times$-bundle over V. The holomorphic fiber coordinate is just $\tau_E + i\phi$, where $\tau_E = i\,\tau$.

Supersymmetric $D3$-brane configurations then correspond to holomorphic surfaces in E with one holomorphic tangent vector along the $\mathbb{C}^\times$ fiber, i.e. any holomorphic surface in E other than V itself. One can verify this statement directly using the kappa-symmetric $D3$-brane worldvolume action [24, 30] as generally illustrated in [31]. In this analysis, the condition for supersymmetry is expressed geometrically as a generalized calibration condition [32, 33] which is satisfied by these holomorphic surfaces in E.

Now let D be a divisor on V. We will consider the supersymmetric $D3$-branes on H such that the projection of the $D3$-brane worldvolume to V is a divisor in the linear equivalence class of D. Let $\mathcal{O}(D)$ be the line-bundle on V associated to D, and let $\pi : E \to V$ denote the projection, so that $\pi^*\mathcal{O}(D)$ is a line-bundle on E. Then the vanishing loci of the global holomorphic sections of $\pi^*\mathcal{O}(D)$ naturally determine holomorphic surfaces in E representing this class of supersymmetric $D3$-brane configurations. The classical configuration space $\mathcal{M}$ is just the projective space associated to the vector space of global holomorphic sections $\mathrm{H}^0(E,\, \pi^*\mathcal{O}(D))$.

4. Quantum supersymmetric $D3$-branes

We finally turn to quantizing the classical configuration space $\mathcal{M}$. Because we are only considering the Hilbert space of BPS states in the brane spectrum, we do not really need any more detailed information about $\mathcal{M}$ other than its complex structure, which we have already described. In particular, although we assume that $\mathcal{M}$ possesses some Kähler metric, the particular form of the metric is mostly irrelevant in the following analysis — a very fortunate thing, since we do not know anything about it. The philosophy we apply falls under the rubric of geometric quantization, for which a general reference is [35].

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For concreteness, we now discuss the examples $H = T^{1,1}$ and $H = S^5$, for which we can compare our results to the dual baryon spectrum.

4.1 Quantum $D3$-branes on $H = T^{1,1}$

In this case, the classical configuration space $\mathcal{M}$ which describes the supersymmetric $D3$-branes wrapped in the A class of H is $\mathbb{CP}^\infty$, with homogeneous coordinates $[c_1 : c_2 : c_{11;1} : \cdots : c_{I;J} : \ldots]$ corresponding to the coefficients of the polynomial in (3.1). We can think of the $D3$-brane as a particle moving on $\mathcal{M}$, and so the $D3$-brane phase space is the cotangent bundle $T^*\mathcal{M}$. The $D3$-brane wavefunction Ψ takes values in some holomorphic line-bundle $\mathcal{L}$ over $\mathcal{M}$, and the hamiltonian which acts on Ψ is the laplacian acting on sections of $\mathcal{L}$. The BPS wavefunctions are the global holomorphic sections of $\mathcal{L}$, and the vector space of these sections is the BPS Hilbert space.

To proceed, we only have to describe the line-bundle $\mathcal{L}$, which is easy since $\mathcal{L}$ is determined by its degree. To compute the degree of $\mathcal{L}$, we consider any $\mathbb{CP}^1$ subspace of $\mathcal{M}$ and a one-parameter family of classical $D3$-brane configurations corresponding to a curve $\mathcal{C}$ in this subspace. If the $D3$-brane configuration changes with time, so that the $D3$-brane traverses the curve $\mathcal{C}$ in $\mathcal{M}$, then Ψ picks up a phase because the $D3$-brane couples to the background RR four-form potential C_4. This phase determines the degree.

Let s be a parameter along $\mathcal{C}$, and $A(s)$ the corresponding family of three-manifolds in the class of A. Finally let D be a disc in $\mathbb{CP}^1$ bounding $\mathcal{C}$. If we take the curve $\mathcal{C}$ in $\mathbb{CP}^1$ to be large, D can be taken to cover the full $\mathbb{CP}^1$ subspace. The corresponding two-parameter family of $D3$-branes sweeps out all of H, and the phase which Ψ acquires when the $D3$-brane traverses the curve $\mathcal{C}$ is

$$2\pi i \oint_{\mathcal{C}} ds \int_{A(s)} C_4 = 2\pi i \int_H F_5 = 2\pi i\, N\,. \tag{4.1}$$

So $\mathcal{L}$ has degree N, i.e. $\mathcal{L} = \mathcal{O}(N)$. In terms of the homogeneous coordinates $[c_1 : c_2 : c_{11;1} : \cdots : c_{I;J} : \cdots]$ of $\mathcal{M}$, the space of global holomorphic sections of $\mathcal{L}$ consists of the degree N polynomials in these coordinates. Thus, the Hilbert space of BPS states is spanned by states of the form

$$|c_{I_1;J_1} \cdots c_{I_N;J_N}\rangle\,. \tag{4.2}$$

At this point, we see that we really need not worry about any subtleties in dealing with holomorphic bundles on infinite-dimensional complex manifolds. Since $\mathcal{L}$ has only finite degree, any given wavefunction Ψ in the BPS sector varies non-trivially only over some finite-dimensional subspace of $\mathcal{M}$. If we were to filter the polynomials in (3.1) by their degree in A_1 and A_2, we would be perfectly justified to consider inductively the corresponding filtration of $\mathcal{M}$ by finite-dimensional subspaces $\mathbb{CP}^1 \subset \mathbb{CP}^7 \subset \cdots \subset \mathcal{M}$ of successively increasing dimension.

We now describe the correspondence between the BPS states in the $D3$-brane Hilbert space and the baryonic operators in the conifold gauge theory. Note that we have an obvious association between the homogeneous coordinates of $\mathcal{M}$ and the chiral operators of $\mathrm{U}(1)_{\mathcal{B}}$ charge one in the gauge theory,

$$c_{i_1 \ldots i_{m+1}; j_1 \ldots j_m} \longleftrightarrow A_{i_1} B_{j_1} A_{i_2} \cdots B_{j_m} A_{i_{m+1}}\,. \tag{4.3}$$

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We may of course permute the elements of $\{i_1 \dots i_{m+1}\}$, $\{j_1 \dots j_m\}$ as we like in the case of the c's, and the F-term equations ensure that the same is true for the chiral operators. So we propose that the correspondence between suitably normalized BPS states and baryons is that

$$|c_{I_1;J_1} \cdots c_{I_N;J_N}\rangle \longleftrightarrow \epsilon_1 \epsilon^2 \left(A_{I_1;J_1}\,,\,\dots\,,\,A_{I_N;J_N}\right). \tag{4.4}$$

Note that both the BPS state and the baryon in (4.4) are fully symmetric in the indices $\{1, 2, \dots, N\}$. This property is a trivial property of degree N monomials and arises due to the anti-symmetrization of gauge indices in the baryon. Also, the global homogeneous coordinates $c_{I;J}$ transform under the induced action of the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ isometry of X and the radial scaling of X exactly as the associated chiral matter fields $A_{I;J}$ transform under the corresponding symmetries of the gauge theory. So the BPS states and baryons in (4.4) have manifestly the same global quantum numbers. Finally, this proposal naturally generalizes the well-known correspondence of the baryonic operators $\mathcal{B}^{(A)}_{i_1 \cdots i_N}$ of least conformal dimension to the BPS states $|c_{i_1} \cdots c_{i_N}\rangle$ which arise from sections of $\mathcal{O}(N)$ over the $\mathbb{CP}^1$ subspace of $\mathcal{M}$ corresponding to the linear polynomials in (3.3).

This quantization procedure directly extends to $D3$-branes wrapped in the classes, such as $2A, 3A, \dots$, of higher degree in H_3. However, as observed in [17], the gauge theory does not seem to possess any new baryons in these sectors beyond those arising from products of baryons in the basic A and B sectors. At this point, we can only venture to speculate that the BPS states which would naively arise from branes wrapped in classes of higher degree on H are susceptible to decay into multi-particle BPS states arising from the branes wrapped in the generating A and B classes.

4.2 Quantum $D3$-branes on $H = S^5$

In this example, the classical configuration space $\mathcal{M}$ consists only of an open subset away from the point $[1:0:0:0:\dots]$ of the $\mathbb{CP}^\infty$ parameter space with homogeneous coordinates $[c : c_i : c_{ij} : c_{ijk} : \cdots : c_I : \cdots]$. The same argument as for $H = T^{1,1}$ implies that the line-bundle $\mathcal{L}$ over $\mathcal{M}$ has degree N and, for wavefunctions which vary non-trivially only over the projective subspace $\mathcal{M}_0 \subset \mathcal{M}$ at $c = 0$, holomorphic sections of $\mathcal{L}$ correspond to degree N polynomials in $c_i, c_{ij}, c_{ijk}, \dots$.

One might worry that, in this case, the non-compactness associated to the coordinate c would lead to many additional wavefunctions, of arbitrary degree in c. However, the essential point is that $\mathcal{M}$ does contain the projective subspace $\mathcal{M}_0$ and can be considered here as the total space of the line-bundle $\mathcal{O}(1)$ over $\mathcal{M}_0$. The twisting in the fiber of this bundle ensures that a section Ψ of $\mathcal{L}$, having at worst a pole of degree N on $\mathcal{M}$, is still a polynomial of at most degree N in c, just as for the homogeneous coordinates on the base $\mathcal{M}_0$.

Thus, despite the fact that the nature of $\mathcal{M}$ is slightly different depending upon whether the $D3$-brane wraps a topologically non-trivial or a trivial three-cycle, in both cases the BPS wavefunctions are degree N polynomials in the global homogeneous coordinates. In the case $H = S^5$, the BPS states are now linear combinations of the states

$$|c_{I_1} \cdots c_{I_N}\rangle\,. \tag{4.5}$$

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To describe the corresponding operators in the $\mathcal{N}=4$ Yang-Mills theory, we decompose the $\mathcal{N}=4$ Yang-Mills multiplet into an $\mathcal{N}=1$ vector multiplet and three $\mathcal{N}=1$ chiral multiplets Z_1, Z_2, Z_3, whose scalar components consist of complex combinations of the six Yang-Mills scalars $\phi_1, \ldots, \phi_6$,

$$z_1 = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\,, \qquad z_2 = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)\,, \qquad z_3 = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)\,. \tag{4.6}$$

We then associate the coordinates c_I with chiral operators as

$$c_I \longleftrightarrow Z_I\,. \tag{4.7}$$

The cubic superpotential that appears when the $\mathcal{N}=4$ theory is decomposed into $\mathcal{N}=1$ representations again ensures that both the left-hand and the right-hand sides of (4.7) are symmetric in the multi-indices I. The novel feature here is that the constant term c in (3.5) is associated to the trivial chiral operator $\mathbf{1}$ in the Yang-Mills theory.

As for $H = T^{1,1}$, we propose the natural correspondence between BPS states and baryonic operators,

$$|c_{I_1} \cdots c_{I_N}\rangle \longleftrightarrow \epsilon_1\epsilon^1\left(Z_{I_1}\,, \,\ldots\,, \,Z_{I_N}\right). \tag{4.8}$$

For instance, we see that the sub-determinant operators

$$\epsilon_1\epsilon^1\left(\overbrace{1,\ldots,1}^{k},\overbrace{Z_1,\ldots,Z_1}^{N-k}\right) \tag{4.9}$$

correspond to BPS states which arise from the affine polynomials $P = c + c_1\, z_1$ classically describing giant gravitons. This observation seems to provide a nice conceptual basis for the proposal of [16].

4.3 Multi-particle states and reducible surfaces

From our description of $\mathcal{M}$, we see that the generic point of $\mathcal{M}$ describes an irreducible surface in X, but there is a closed subset $\mathcal{M}_R$ which consists of reducible surfaces. Only the irreducible surfaces classically describe single branes, and the reducible surfaces describe configurations of multiple branes. So when we quantize $\mathcal{M}$, some of the wavefunctions we obtain describe single-particle states, and others must describe multi-particle states.

But which of our BPS states are the single-particle states, and which are the multi-particle states? The answer to this question must involve the behavior of the Kähler metric on $\mathcal{M}$, since only the single-particle wavefunctions are normalizable in L^2 on $\mathcal{M}$. A reasonable hypothesis, based on the classical observations above, is that this metric is such that the reducible locus $\mathcal{M}_R$ lies at infinite distance from all other points in $\mathcal{M} - \mathcal{M}_R$.

If this guess is correct, then any single-particle wavefunction on $\mathcal{M}$, being normalizable, would necessarily vanish over the locus $\mathcal{M}_R$, as the classical reasoning suggests. Conversely, wavefunctions on $\mathcal{M}$ not vanishing over $\mathcal{M}_R$, i.e. extending to "infinity", would naturally be interpretable as non-normalizable, multi-particle states.

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To describe how this idea can be interpreted in a simple case, for which X is the conifold, we consider the locus in $\mathcal{M}_R$ corresponding to factorizable polynomials of the form

$$P(A_1, A_2, B_1, B_2) = (a_1 A_1 + a_2 A_2) \cdot \left(\sum_{I;J} b_{I;J}\, A^I B^J \right), \tag{4.10}$$

where the second factor is a general polynomial of charge zero. The coefficients $[a_1 : a_2]$ and $[b : b_{1;1} : b_{2;1} : \cdots : b_{I;J} : \cdots]$ determine homogeneous coordinates on a $\mathbb{CP}^1 \times \mathbb{CP}^\infty$ family[3] of such reducible surfaces in X. There is a natural map $f : \mathbb{CP}^1 \times \mathbb{CP}^\infty \to \mathcal{M}$ from this family of reducible surfaces into the general space of holomorphic surfaces $\mathcal{M}$. This map just corresponds to multiplying out the coefficients in (4.10), so in terms of the homogeneous coordinates $c_{I;J}$ of $\mathcal{M}$, f is expressed by

$$c_{i_1 i_2 \cdots i_{m+1}; j_1 \cdots j_m} = a_{(i_1} \cdot b_{i_2 \cdots i_{m+1}); j_1 \cdots j_m}\,, \tag{4.11}$$

where we symmetrize over the indices $\{i_1, \ldots, i_{m+1}\}$ appearing on the right-hand side of (4.11). Under f, sections of $\mathcal{O}(N)$ on $\mathcal{M}$ pull-back to sections of $\mathcal{O}(N) \otimes \mathcal{O}(N)$ on $\mathbb{CP}^1 \times \mathbb{CP}^\infty$.

We now consider a general element of the Hilbert space of the form

$$t_{R;S}^{I_1 \cdots I_N; J_1 \cdots J_N} |c_{I_1;J_1} \cdots c_{I_N;J_N}\rangle\,, \tag{4.12}$$

where $t_{R;S}^{I_1 \cdots I_N; J_1 \cdots J_N}$ is the same tensor used to make the general baryon $\mathcal{B}^{(A)}_{R;S}$ in (2.28). Upon pull-back,

$$f^* \left(t_{R;S}^{I_1 \cdots I_N; J_1 \cdots J_N} |c_{I_1;J_1} \cdots c_{I_N;J_N}\rangle \right) = t_{R;S}^{I_1 \cdots I_N; J_1 \cdots J_N} |a_{i_1} \, \cdots \, a_{i_N}\rangle \otimes |b_{K_1;J_1} \, \cdots \, b_{K_N;J_N}\rangle\,, \tag{4.13}$$

where our notation is as for the factorizable baryon in (2.30).

We now make essentially the same observation which we made for the baryons in section 2.3. The tensor product of states on the right-hand side of (4.13) necessarily transforms in the representation which is fully-symmetric on SU(2) A flavor indices — so the pull-back of the general state in (4.12) to this reducible locus vanishes unless R is the fully-symmetric representation.

Thus our guess about the metric on $\mathcal{M}$ implies that a state of the form (4.12) for which R is the fully-symmetric representation must be a multi-particle state. As we have seen in section 2.3, the corresponding baryon always factors into a product of the minimal baryon $\mathcal{B}^{(A)}_{i_1 \cdots i_N}$ and some other baryon $\mathcal{B}^{(0)}_{K_1 \cdots K_N; J_1 \cdots J_N}$. Under our general proposal relating BPS states and baryons, we naturally identify this product of baryons as corresponding to the tensor product of states in (4.13).

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[3] More precisely, we should remove the point of the $\mathbb{CP}^\infty$ factor corresponding to the constant polynomial, so that we only consider non-trivial factorizations of P.

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